Jacobians of modular curves associated to normalizers of Cartan subgroups of level p^n

Jacobiennes de courbes modulaires associées aux normalisateurs de sous-groupes de Cartan de niveau p^n

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Abstract

We derive a relation between induced representations on the group $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ which implies a relation between the Jacobians of certain modular curves of level p^n. A consequence of this relation is that the Jacobian of the modular curve associated to the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ does not have any non-zero rank 0 quotients defined over $\mathbb{Q}$ if the Birch and Swinnerton-Dyer conjecture holds for Abelian varieties.

Résumé

Nous établissons une relation entre des représentations induites sur le groupe $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, ce qui implique une relation entre les jacobiennes de certaines courbes modulaires de niveau p^n. Une conséquence de cette relation est que la jacobienne de la courbe modulaire associée au normalisateur d'un sous-groupe Cartan non-déployé de $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ n'a aucun quotient non-nuls de rang 0 defini sur $\mathbb{Q}$ si l'on admet la conjecture de Birch et Swinnerton-Dyer pour les variétés abéliennes.

1. Introduction

Let p be an odd prime and $n \in \mathbb{N}$. Let $R = \mathbb{Z}/p^n\mathbb{Z}$ and $G = \mathrm{GL}_2(R)$. Consider the subgroups N', B_{s-1}, N, T_r of G described explicitly in Table 1 where by convention we let $T_0 = G$. These subgroups can be described in the following manner. Let G act on $\mathbb{P}^1(S)$ from the left where $S = R[Y]/(Y^2 - \epsilon)$ and ϵ is a non-square in $R^\times$. The subgroups N' and N are stabilizers in G of the subsets $\{Y, -Y\}$ and

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$\{0,\infty\}$ respectively. Let G act on $\mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z}) \times \mathbb{P}^1(\mathbb{Z}/p^r\mathbb{Z})$ diagonally and by reduction modulo p^r on each component. The subgroup T_r is the stabilizer in G of $(0,\infty)$. Let G act on $\mathbb{P}^1(\mathbb{Z}/p^{s-1}\mathbb{Z}) \times \mathbb{P}^1(\mathbb{Z}/p^s\mathbb{Z})$ diagonally and by reduction modulo p^{s-1} and modulo p^s respectively. The subgroup B_{s-1} is the stabilizer in G of $(0,\infty)$. In the case $n = 1$, N' and N are the normalizers of non-split and split Cartan subgroups of G respectively, T_1 is a split Cartan subgroup of G, and B_0 is a Borel subgroup of G.

For a subgroup H of G, let $\mathrm{Ind}_H^G 1$ be the induction of the trivial representation of H to G where representations are assumed to act on $\mathbb{Q}$-vector spaces. If two representations ρ_1 and ρ_2 of G are isomorphic over $\mathbb{Q}$ (i.e. their representation spaces are isomorphic as $\mathbb{Q}[G]$-modules), we write $\rho_1 \cong_{\mathbb{Q}} \rho_2$.

Theorem 1.1 *For each $n \in \mathbb{N}$, we have that:*

$$\mathrm{Ind}_{N'}^G 1 \oplus \bigoplus_{s=1}^{n} \mathrm{Ind}_{B_{s-1}}^G 1 \cong_{\mathbb{Q}} \mathrm{Ind}_N^G 1 \oplus \bigoplus_{r=0}^{n-1} \mathrm{Ind}_{T_r}^G 1. \tag{1}$$

Proof. – Tables 2 and 3 describe the conjugacy classes and related information of the group G. This can be used to compute the character values in Table 3 and the first two columns of Table 4. By reduction modulo lower powers of p, we may deduce the last two columns of Table 4. The character values listed in Table 3 and Table 4 allow us to verify that the character of the representation on the left hand side of (1) is equal to the character of the representation on right hand side (1), thereby showing the two representations in question are isomorphic over $\mathbb{Q}$.

Let $X(p^n)$ denote the compactified modular curve classifying elliptic curves with full level p^n structure [8]. By a full level p^n structure of an elliptic curve E over a scheme S, we mean a group homomorphism ϕ from $A = \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ to $E[p^n](S)$ such that $\sum_{a\in A}[\phi(a)]$ is equal to $E[p^n]$ as a Cartier divisor of E over S where $[\phi(a)]$ denotes the Cartier divisor associated to $\phi(a)$. This modular curve has a model over $\mathbb{Q}$ which is geometrically disconnected and which has a right group action of G also defined over $\mathbb{Q}$. For a subgroup H of G, let $X_H(p^n)$ be the quotient of $X(p^n)$ by H and $J_H(p^n)$ be its Jacobian (taken to be its Picard variety). If two Abelian varieties A_1 and A_2 defined over $\mathbb{Q}$ are isogenous over $\mathbb{Q}$ we write $A_1 \sim_{\mathbb{Q}} A_2$. If they are isomorphic over $\mathbb{Q}$, we write $A_1 \cong_{\mathbb{Q}} A_2$.

Theorem 1.2 *For each $n \in \mathbb{N}$, we have that:*

$$J_{N'}(p^n) \times \prod_{s=1}^{n} J_{B_{s-1}}(p^n) \sim_{\mathbb{Q}} J_N(p^n) \times \prod_{r=0}^{n-1} J_{T_r}(p^n).$$

Proof. – Using the general method in [6], we deduce the theorem from the relation of induced representations in Theorem 1.1. This generalizes the case $n = 2$ shown in [4].

For a non-negative integer r, let $X_0^+(p^r)$ denote the quotient of the modular curve $X_0(p^r)$ by its Fricke involution W_{p^r} where by convention we let W_{p^r} be the identity and $X_0^+(p^r) = X_0(p^r)$ if $r = 0$. Let $J_0(p^r)$ and $J_0^+(p^r)$ denote the Jacobians of the modular curves $X_0(p^r)$ and $X_0^+(p^r)$ respectively. Let $N_0(p^t)$ and $N_0^+(p^t)$ denote the new quotients of $J_0(p^t)$ and $J_0^+(p^t)$ respectively, defined as the quotients by the sums of images of degeneracy morphisms from lower levels (cf. [10] [12]).

Proposition 1.3 *We have that:*

$$J_0(p^r) \sim_{\mathbb{Q}} \prod_{t=0}^{r} N_0(p^t)^{r-t+1} \text{ and } J_0^+(p^r) \sim_{\mathbb{Q}} \prod_{t=0}^{r} N_0(p^t)^{\frac{r-t+1}{2}}$$

where by convention we let

$$N_0(p^t)^{\frac{m}{2}} = \begin{cases} N_0(p^t)^{\frac{m}{2}} \text{ if } m \text{ is even} \\ N_0(p^t)^{\frac{m-1}{2}} \times N_0^+(p^t) \text{ if } m \text{ is odd.} \end{cases}$$

Proof. – One can construct a homomorphism from the left hand side to the products on the right hand side of each relation using degeneracy morphisms. It suffices to verify that the induced map on the corresponding spaces of cusp forms of weight 2 is an isomorphism. This can be shown using the results of Atkin-Lehner theory [1], Theorem 5 and Lemma 26.

Using the facts that

$$J_N(p^n) \cong_{\mathbb{Q}} J_0^+(p^{2n}),$$
$$J_{T_r}(p^n) \cong_{\mathbb{Q}} J_0(p^{2r}),$$
$$J_{B_{s-1}}(p^n) \cong_{\mathbb{Q}} J_0(p^{2s-1}),$$

which can be obtained from results in [8] or [16], and Theorem 1.2, we deduce that

$$J_{N'}(p^n) \times \prod_{s=1}^{n} J_0(p^{2s-1}) \sim_{\mathbb{Q}} J_0^+(p^{2n}) \times \prod_{r=0}^{n-1} J_0(p^{2r}). \tag{2}$$

Theorem 1.4 *For each $n \in \mathbb{N}$, we have that:*

$$J_{N'}(p^n) \sim_{\mathbb{Q}} \prod_{r=0}^{n} N_0^+(p^{2r}).$$

Proof. – This can be shown by counting the number of copies of $N_0(p^t)$ up to isogeny over $\mathbb{Q}$ on both sides of (2) using Proposition 1.3.

We remark that the case $n = 1$ and variants of it with additional level structure have been known to experts for some time (cf. reference to Ligozat in [7] and Elkies in [5]). More recent references in the literature include [2] [6] [14]. We also note that the special case $N^+ = 1, N^- = p^{2n}$ of Corollary 3.3.2 in [13], derived by means of trace formulae and Faltings' isogeny theorem, is a variant of Theorem 1.4 where one considers Cartan subgroups rather than the normalizers of Cartan subgroups N' and N.

Theorem 1.5 *Suppose that the Birch and Swinnerton-Dyer conjecture holds for Abelian varieties. Then for each $n \in \mathbb{N}$, the Abelian variety $J_{N'}(p^n)$ has no non-zero rank 0 quotients defined over $\mathbb{Q}$.*

Proof. – The L-functions of simple factors of $N_0^+(p^{2r})$ defined over $\mathbb{Q}$ are forced to vanish at $s = 1$ by consideration of signs in functional equations. Hence, every simple factor defined over $\mathbb{Q}$ of $N_0^+(p^{2r})$ has positive rank over $\mathbb{Q}$ by the Birch and Swinnerton-Dyer conjecture.

It has been known for some time that the modular curve $X_{N'}(p)$ represents the most difficult case of Serre's question on the surjectivity of Galois representations associated to elliptic curves [15] [9]. The results above show that this difficulty does not disappear when the level p is replaced by a power of p.

It would be interesting to determine an explicit description of the isogeny in Theorem 1.2. In the case $n = 1$, an explicit description was conjectured by Merel [11] and subsequently proven in [3].

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Table 1
Conventions and definitions.

conventions used in tables
p is an odd prime
$R = \mathbb{Z}/p^n\mathbb{Z}$
$\epsilon \in R^\times$ is a non-square
$m = n - 1$
$1 \le r, s \le n$
$1 \le \mu < \nu \le n - 1$
$[\cdot]$ denotes the value 1 if $\cdot$ is true and 0 otherwise
t denotes the trace of the conjugacy class

subgroup	form of elements	order
N'	$\left\{ \begin{pmatrix} a & b\epsilon \\ b & a \end{pmatrix}, \begin{pmatrix} a & b\epsilon \\ -b & -a \end{pmatrix} \right\}$	$2p^{2m} \cdot (p^2 - 1)$
B_{s-1}	$\left\{ \begin{pmatrix} a & bp^{s-1} \\ cp^s & d \end{pmatrix} \right\}$	$p^{4n-2s-1} \cdot (p-1)^2$
N	$\left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}$	$2p^m \cdot (p-1)^2$
T_r	$\left\{ \begin{pmatrix} a & bp^r \\ cp^r & d \end{pmatrix} \right\}$	$p^{4n-2r-2} \cdot (p-1)^2$

Table 2
Conjugacy classes of G.

type	representatives	parameters	form of elements in centralizer
I	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\alpha \in R^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$
T'	$\begin{pmatrix} \alpha & \epsilon\beta^2 \\ 1 & \alpha \end{pmatrix}$	$\alpha \in R, \beta \in R^\times$	$\left\{ \begin{pmatrix} a & c\epsilon\beta^2 \\ c & a \end{pmatrix} \right\}$
B	$\begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}$	$\alpha \in R^\times$	$\left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \right\}$
T	$\begin{pmatrix} \alpha & \beta^2 \\ 1 & \alpha \end{pmatrix}$	$\alpha \in R, \beta \in R^\times$	$\left\{ \begin{pmatrix} a & c\beta^2 \\ c & a \end{pmatrix} \right\}$
RT'_ν	$\begin{pmatrix} \alpha & p^\nu\epsilon\beta^2 \\ 1 & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\nu}R)^\times$	$\left\{ \begin{pmatrix} a & cp^\nu\epsilon\beta^2 \\ c & a \end{pmatrix} \right\}$
RT_ν	$\begin{pmatrix} \alpha & p^\nu\beta^2 \\ 1 & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\nu}R)^\times$	$\left\{ \begin{pmatrix} a & cp^\nu\beta^2 \\ c & a \end{pmatrix} \right\}$
RI'_μ	$\begin{pmatrix} \alpha & p^\mu\epsilon\beta^2 \\ p^\mu & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\mu}R)^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \pmod{p^{n-\mu}}, b \equiv c\epsilon\beta^2 \pmod{p^{n-\mu}} \right\}$
$RBI'_{\mu,\nu}$	$\begin{pmatrix} \alpha & p^\nu\epsilon\beta^2 \\ p^\mu & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\nu}R)^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \pmod{p^{n-\mu}}, b \equiv cp^{\nu-\mu}\epsilon\beta^2 \pmod{p^{n-\mu}} \right\}$
RB_μ	$\begin{pmatrix} \alpha & 0 \\ p^\mu & \alpha \end{pmatrix}$	$\alpha \in R^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \pmod{p^{n-\mu}}, b \equiv 0 \pmod{p^{n-\mu}} \right\}$
$RBI_{\mu,\nu}$	$\begin{pmatrix} \alpha & p^\nu\beta^2 \\ p^\mu & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\nu}R)^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \pmod{p^{n-\mu}}, b \equiv cp^{\nu-\mu}\beta^2 \pmod{p^{n-\mu}} \right\}$
RI_μ	$\begin{pmatrix} \alpha & p^\mu\beta^2 \\ p^\mu & \alpha \end{pmatrix}$	$\alpha \in R^\times, \beta \in (R/p^{n-\mu}R)^\times$	$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \pmod{p^{n-\mu}}, b \equiv c\beta^2 \pmod{p^{n-\mu}} \right\}$

Table 3
Conjugacy information and values of the characters of some induced representations of G.

type	number of this type	size of centralizer	size of conjugacy class	$\mathrm{Ind}_{N'}^G 1$	$\mathrm{Ind}_N^G 1$
I	$p^m(p-1)$	$p^{4m}\cdot(p^2-1)(p^2-p)$	1	$\frac{p^{2m}(p^2-p)}{2}$	$\frac{p^{2m}(p^2+p)}{2}$
$T'(t=0)$	$p^m(p-1)/2$	$p^{2m}\cdot(p^2-1)$	$p^{2m}\cdot(p^2-p)$	$1+\frac{p^m(p+1)}{2}$	$\frac{p^m(p+1)}{2}$
$T'(t\neq 0)$	$(p^n-1)\cdot p^m(p-1)/2$	$p^{2m}\cdot(p^2-1)$	$p^{2m}\cdot(p^2-p)$	1	0
B	$p^m(p-1)$	$p^m(p-1)\cdot p^n$	$p^{2m}\cdot(p^2-1)$	0	0
$T(t=0)$	$p^m(p-1)/2$	$p^{2m}(p-1)^2$	$p^{2m}\cdot(p^2+p)$	$\frac{p^m(p-1)}{2}$	$1+\frac{p^m(p-1)}{2}$
$T(t\neq 0)$	$(p^n-2p^m-1)\cdot p^m(p-1)/2$	$p^{2m}(p-1)^2$	$p^{2m}\cdot(p^2+p)$	0	1
RT'_ν	$p^m(p-1)\cdot p^{m-\nu}(p-1)/2$	$p^m(p-1)\cdot p^n$	$p^{2m}\cdot(p^2-1)$	0	0
RT_ν	$p^m(p-1)\cdot p^{m-\nu}(p-1)/2$	$p^m(p-1)\cdot p^n$	$p^{2m}\cdot(p^2-1)$	0	0
RI'_μ	$p^m(p-1)\cdot p^{m-\mu}(p-1)/2$	$p^{2m+2\mu}\cdot(p^2-1)$	$p^{2m-2\mu}\cdot(p^2-p)$	$p^{2\mu}$	0
$RBI'_{\mu,\nu}$	$p^m(p-1)\cdot p^{m-\nu}(p-1)/2$	$p^{2m+2\mu}\cdot p(p-1)$	$p^{2m-2\mu}\cdot(p^2-1)$	0	0
RB_μ	$p^m(p-1)$	$p^{2m+2\mu}\cdot p(p-1)$	$p^{2m-2\mu}\cdot(p^2-1)$	0	0
$RBI_{\mu,\nu}$	$p^m(p-1)\cdot p^{m-\nu}(p-1)/2$	$p^{2m+2\mu}\cdot p(p-1)$	$p^{2m-2\mu}\cdot(p^2-1)$	0	0
RI_μ	$p^m(p-1)\cdot p^{m-\mu}(p-1)/2$	$p^{2m+2\mu}\cdot(p-1)^2$	$p^{2m-2\mu}\cdot(p^2+p)$	0	$p^{2\mu}$

Table 4
Values of the characters of some induced representations of G.

type	$\mathrm{Ind}_{B_{n-1}}^G 1$	$\mathrm{Ind}_{T_n}^G 1$	$\mathrm{Ind}_{B_{s-1}}^G 1$	$\mathrm{Ind}_{T_r}^G 1$
I	$p^{2m}\cdot(p+1)$	$p^{2m}\cdot p(p+1)$	$p^{2(s-1)}\cdot(p+1)$	$p^{2(r-1)}\cdot p(p+1)$
$T'(t=0)$	0	0	0	0
$T'(t\neq 0)$	0	0	0	0
B	$1\cdot[n=1]$	0	$1\cdot[s=1]$	0
$T(t=0)$	2	2	2	2
$T(t\neq 0)$	2	2	2	2
RT'_ν	0	0	$1\cdot[s=1]$	0
RT_ν	0	0	$1\cdot[s=1]$	0
RI'_μ	0	0	$p^{2(s-1)}\cdot(p+1)\cdot[\mu\geq s]$	$p^{2(r-1)}\cdot p(p+1)\cdot[\mu\geq r]$
$RBI'_{\mu,\nu}$	0	0	$p^{2(s-1)}\cdot(p+1)\cdot[\mu\geq s]+p^{2(s-1)}\cdot[\mu=s-1]$	$p^{2(r-1)}\cdot p(p+1)\cdot[\mu\geq r]$
RB_μ	$p^{2m}\cdot[\mu=m]$	0	$p^{2(s-1)}\cdot(p+1)\cdot[\mu\geq s]+p^{2(s-1)}\cdot[\mu=s-1]$	$p^{2(r-1)}\cdot p(p+1)\cdot[\mu\geq r]$
$RBI_{\mu,\nu}$	0	0	$p^{2(s-1)}\cdot(p+1)\cdot[\mu\geq s]+p^{2(s-1)}\cdot[\mu=s-1]$	$p^{2(r-1)}\cdot p(p+1)\cdot[\mu\geq r]$
RI_μ	$2p^{2\mu}$	$2p^{2\mu}$	$p^{2(s-1)}\cdot(p+1)\cdot[\mu\geq s]+2p^{2\mu}\cdot[\mu<s]$	$p^{2(r-1)}\cdot p(p+1)\cdot[\mu\geq r]+2p^{2\mu}\cdot[\mu<r]$