COEFFICIENTS OF EXPONENTIAL FUNCTIONS ATTACHED TO DRINFELD MODULES OF RANK 2

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Abstract. Let ϕ be a Drinfeld A-module of rank 2 defined over C_∞. We explicitly determine the pattern of valuations of the exponential functions attached to ϕ and discuss applications to the study of zeroes of para-Eisenstein series.

1. Introduction

Let $K = \mathbb{F}_q(T)$, $A = \mathbb{F}_q[T]$, and let $\infty = (\frac{1}{T})$ be the place at infinity of K with associated valuation function $v = v_\infty : K \to \mathbb{Z} \cup \{\infty\}$ so that $v_\infty(f/g) = \deg g - \deg f$, for $f, g \in A$. Then $K_\infty = \mathbb{F}_q((\frac{1}{T}))$. Let C_∞ be the completion of an algebraic closure of K_∞ and denote also by v the extension of v from K to C_∞. Let the absolute value associated to v be given by $|x| = q^{-v(x)}$.

Let L be an extension of K. A *Drinfeld A-module ϕ over L* is an $\mathbb{F}_q$-algebra homomorphism

$$\phi : A \to L\langle\tau\rangle, \quad a \mapsto \phi_a := \sum_{i=0}^{m_{\phi,a}} a_i(\phi, a)\tau^i$$

such that for any a in A, ϕ_a has constant term a, where $\tau : x \mapsto x^q$ is the q-th power Frobenius endomorphism, and $L\langle\tau\rangle$ denotes the ring of twisted polynomials over L, with $\tau\alpha = \alpha^q\tau$ for all $\alpha \in L$. We require that the image of ϕ not be contained in L. There is a unique positive integer r such that $\deg_\tau \phi_a = r \deg a$ for all $a \in A$, and such an r is defined to the *rank* of a Drinfeld A-module ϕ. We note that a Drinfeld module ϕ of rank 2 defined over L is completely determined by $\phi_T = T + a_1\tau + a_2\tau^2$, where $a_i = a_i(\phi) \in L$ and $a_2 \neq 0$.

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Let ϕ be a Drinfeld A-module of rank 2 over C_∞. Let $j(\phi)$ be the *j-invariant* of ϕ, defined by $j(\phi) = a_1(\phi)^{q+1}/a_2(\phi)$. By uniformization (cf. [6]), there is an A-lattice $\Lambda_\phi = \Lambda_{\phi,\infty} \subseteq C_\infty$ of rank 2 and a surjective analytic function $e_\phi = e_{\phi,\infty} : C_\infty \to C_\infty$ with zero set equal to Λ_ϕ such that

$$e_\phi(az) = \phi_a \circ e_\phi(z) \tag{1}$$

for all $a \in A$, and e_ϕ is normalized so $e'_\phi(z) = 1$. The function $e_\phi(z)$ is called the *exponential function* attached to ϕ. It is uniquely determined by the above properties and can be written in the form $e_\phi(z) = \sum_{i=0}^{\infty} c_i \tau^i(z)$ where $\tau(z) = z^q$, $c_i \in C_\infty$, and $c_0 = 1$.

In this paper, we determine in many cases the valuations of the coefficients of e_ϕ in terms of the defining coefficients of $\phi_T = T + a_1\tau + a_2\tau^2$. Our methods are elementary and algebraic, but nonetheless reveal some interesting closed form patterns and results not covered by more conceptual methods in the literature. For a more detailed discussion, see Section 4 where we put our results in context.

Finally, we make the remark that determining the valuations of exponential functions is a natural problem. For instance, this is well known for the classical complex exponential function e^z as well as the exponential function of a rank 1 Drinfeld A-module. Our paper is in some sense a generalization of these computations to a setting where there is a whole family of exponential functions.

2. Recursion formulae for the coefficients of exponential functions

Let $e_\phi = \sum_{i=0}^{\infty} c_i \tau^i$ be the exponential function associated to a Drinfeld module of rank 2 over C_∞, given by $\phi_T = T + a_1\tau + a_2\tau^2$. The exponential function is normalized so that $c_0 = 1$ and the following formulae determine its coefficients [6] (we take $a_0 = T$ in Lemma 4.6.5).

$$c_1 = \frac{a_1 c_0^q}{T^q - T}, \tag{2}$$

$$c_i = \frac{a_1 c_{i-1}^q + a_2 c_{i-2}^{q^2}}{T^{q^i} - T} \text{ for } i \geq 2. \tag{3}$$

Let $d_i = \frac{v(c_i)}{q^i}$. Then we have the following formulae for the d_i's.

$$d_0 = 0,$$

$$d_1 = \left(\frac{v(a_1)}{q} + 1\right) + d_0,$$

$$d_i \geq \min\left(\frac{v(a_1)}{q^i} + d_{i-1}, \frac{v(a_2)}{q^i} + d_{i-2}\right) + 1 \text{ for } i \geq 2,$$

where equality holds if the values in the minimum are distinct.

Lemma 2.1. *If $a_1 = 0$, then $c_i = 0$ when i is odd, and c_i is nonzero when i is even. On the other hand, if a_1 is nonzero, then it can never happen that two of c_i, c_{i+1}, c_{i+2} vanish. That is to say, it does not happen that two of d_i, d_{i+1}, d_{i+2} are infinite.*

Proof. We first note that $a_1 = 0$ if and only if $c_1 = 0$. If $a_1 = 0$, then it follows from the recursion formulae (2) that c_i is nonzero when i is even and $c_i = 0$ when i is odd.

If a_1 is nonzero, then c_1 is nonzero. If we assume that two consecutive c_i, c_{i+1} vanish simultaneously, then it follows immediately from the recursion (2) that c_{i-1} also vanishes. Repeating this argument, we arrive at $c_1 = 0$, a contradiction. Now if we assume that $c_i = 0, c_{i+1} \neq 0, c_{i+2} = 0$, then from $a_1 c_{i+1}^q + a_2 c_i^{q^2} = 0$, we get $c_{i+1} = 0$, a contradiction. □

If $\frac{v(a_1)}{q^i} + d_{i-1} \neq \frac{v(a_2)}{q^i} + d_{i-2}$, then we note that at the ith step in the recursion sequence, there are two choices: the new term d_i to be computed is either derived by a formula involving the previous term, or the term before the previous term. For $i \geq 1$, we say that d_i is of *type* I, II, or III, if $\frac{v(a_1)}{q^i} + d_{i-1} < \frac{v(a_2)}{q^i} + d_{i-2}$, $\frac{v(a_1)}{q^i} + d_{i-1} > \frac{v(a_2)}{q^i} + d_{i-2}$, or $\frac{v(a_1)}{q^i} + d_{i-1} = \frac{v(a_2)}{q^i} + d_{i-2}$, respectively. If d_i is of type I, then $d_i = \frac{v(a_1)}{q^i} + d_{i-1} + 1$; if d_i is of type II, then $d_i = \frac{v(a_2)}{q^i} + d_{i-2} + 1$.

Remark 2.2. *Suppose we have an isomorphism f from $\phi_T = T + a_1\tau + a_2\tau^2$ to $\phi'_T = T + a'_1\tau + a'_2\tau^2$. It follows that $f \circ \phi_a = \phi'_a \circ f$ for all $a \in A$, $f(z) = cz$ for some $c \in C_\infty^*$, and $a'_i = a_i/c^{q^i-1}$. Using Equation (1), we see by induction that $e_{\phi'}(z) = \sum_{i=0}^{\infty} c'_i \tau^i(z) = \sum_{i=0}^{\infty} \frac{c_i}{c^{q^i-1}} \tau^i(z)$. It follows that $f \circ e_\phi = e_{\phi'} \circ f$ and hence $\Lambda_{\phi'} = c\Lambda_\phi$.*

Let $d_i = v(c_i)/q^i$ and $d'_i = v(c'_i)/q^i$. Then one can see using $c'_i = c_i/c^{q^i-1}$ and $a'_i = a_i/c^{q^i-1}$ that the types of d_i and d'_i are the same for each $i \geq 1$. Hence, the pattern of types of d_i is an invariant of the C_∞-isomorphism class of ϕ.

A *run of type* I (or type II) is a subsequence of consecutive terms in the recursion sequence all of which are type I (or type II).

Our strategy is to regard the sequence as being grouped into runs: starting with a run of type I, then a run of type II, then a run of type I, etc. With this point of view, we determine the exact conditions which tell us when we switch from a run of one type to a run of another type.

First note that d_1 is of type I, by definition. Thus, one always begins with a run of a type I, that is $d_i = \left(\frac{v(a_1)}{q^i} + 1\right) + d_{i-1}$ starting from $i = 1$. When do we switch over in the sense that d_{i+1} has type II or III ? The following lemma answers this question.

Lemma 2.3. *Let m be the smallest integer $m \geq 1$ such that $-v(j(\phi)) \leq q^{m+1}$. Then*

(i) $d_1, d_2, \ldots, d_m$ *is a run of type I.*
(ii) d_{m+1} *is of type* II *(resp.* III*) if* $-v(j(\phi)) < q^{m+1}$ *(resp.* $-v(j(\phi)) = q^{m+1}$*).*

Proof. We begin with a run of a type I, that is $d_i = \left(\frac{v(a_1)}{q^i} + 1\right) + d_{i-1}$ starting from $i = 1$. We observe that the type of d_{i+1} is II or III if

$$\frac{v(a_1)}{q^{i+1}} + d_i \geq \frac{v(a_2)}{q^{i+1}} + d_{i-1},$$

equivalently if

$$\frac{v(a_1)}{q^{i+1}} + \frac{v(a_1)}{q^i} + 1 \geq \frac{v(a_2)}{q^{i+1}}. \tag{4}$$

We also note that (4) is equivalent to

$$\frac{v(a_2)}{q+1} - v(a_1) \leq \frac{q^{i+1}}{q+1},$$

which is equivalent to $-v(j(\phi)) \leq q^{i+1}$. Therefore, if m is the least integer $m \geq 1$ such that

$$-v(j(\phi)) \leq q^{m+1}, \tag{5}$$

then $d_1, \ldots, d_m$ is a run of type I and d_{m+1} is of type II (resp. III) if we have strict inequality (resp. equality) in (5). ☐

We note that once (4) holds for $i = m$, it will hold for any $i \geq m$ (see Proposition 2.6, part (i)).

If d_i is of type III, let $\delta_i \geq 0$ be defined as:

$$\begin{aligned} d_i &= d_{i-1} + \frac{v(a_1)}{q^i} + 1 + \delta_i \\ &= d_{i-2} + \frac{v(a_2)}{q^i} + 1 + \delta_i, \end{aligned} \tag{6}$$

where δ_i may be ∞, and $\delta_i = 0$ when $d_i = d_{i-1} = \infty$ or $d_i = d_{i-2} = \infty$.

If d_i is not of type III, let $\delta_i = 0$.

Throughout this paper for the convenience of notation, we set

$$\mathcal{D}_i := \left(\frac{v(a_1)}{q^{i+1}} + d_i\right) - \left(\frac{v(a_2)}{q^{i+1}} + d_{i-1}\right).$$

Lemma 2.4. *Let m be the smallest integer $m \geq 1$ such that $-v(j(\phi)) \leq q^{m+1}$. Suppose that d_j is of type* III *and $d_{j+1}, d_{j+2}, \ldots, d_{j+i}$ is a run of type* II *for some $j \geq m+1$ and $i \geq 1$. Let*

$$e_{ij} = \begin{cases} \frac{q^{i+j+1}(1+\delta_j)}{q^{i+1}+1} & \text{if } i \text{ is even,} \\ \frac{q^{i+j+1}\delta_j}{q^{i+1}-1} & \text{if } i \text{ is odd.} \end{cases}$$

Then d_{j+i+1} is of type:

I *when i is even and $-v(j(\phi)) > (q+1)e_{ij}$, or when i is odd and $-v(j(\phi)) < (q+1)e_{ij}$.*
II *when i is even and $-v(j(\phi)) < (q+1)e_{ij}$, or when i is odd and $-v(j(\phi)) > (q+1)e_{ij}$.*
III *when $-v(j(\phi)) = (q+1)e_{ij}$.*

Proof. Assume that d_j is of type III and $d_{j+1}, d_{j+2}, \ldots, d_{j+i}$ is a run of type II, for some $j \geq 0$ and $i \geq 0$. Then $d_{j+k} = \frac{v(a_2)}{q^{j+k}} + d_{j+k-2} + 1$ for $1 \leq k \leq i$.

Assume that i is even, then we have

$$\begin{aligned} \mathcal{D}_{j+i} &= \frac{v(a_1) - v(a_2)}{q^{j+i+1}} + (d_{j+i} - d_{j+i-1}) \\ &= \frac{v(a_1) - v(a_2)}{q^{j+i+1}} + \frac{qv(a_2) - q^2 v(a_2)}{q^{j+i+1}} + (d_{j+i-2} - d_{j+i-3}) \\ &\vdots \\ &= \frac{1}{q^{j+i+1}} \left(v(a_1) + v(a_2)(-1 + (q - q^2) + (q^3 - q^4) + \ldots + (q^{i-1} - q^i))\right) \\ &\quad + (d_{j+i-i} - d_{j+i-i-1}) \\ &= \frac{1}{q^{j+i+1}} \left(v(a_1) + v(a_2)\frac{(-q)^{i+1} - 1}{q+1}\right) + \left(\frac{v(a_1)}{q^{j+i-i}} + 1 + \delta_{j+i-i}\right) \\ &= \frac{1}{q^{j+i+1}} \left(v(a_1)(1 + q^{i+1}) + v(a_2)\frac{(-q)^{i+1} - 1}{q+1} + q^{j+i+1}(1 + \delta_{j+i-i})\right) \\ &= \frac{q^{i+1} + 1}{q^{j+i+1}} \left(-\frac{v(a_2)}{q+1} + v(a_1) + \frac{q^{j+i+1}}{q^{i+1}+1}(1 + \delta_{j+i-i})\right), \end{aligned}$$

so d_{j+i+1} has type I if and only if $\mathcal{D}_{j+i} < 0$, which is if and only if $\frac{v(a_2)}{q+1} - v(a_1) > \frac{q^{j+i+1}(1+\delta_{j+i-i})}{q^{i+1}+1}$. The other cases follow immediately.

When i is odd, in a similar way we have

$$\begin{aligned}
\mathcal{D}_{j+i} &= \frac{1}{q^{j+i+1}}\left(v(a_1)+v(a_2)(-1+q-q^2+\ldots-q^{i-1})\right)+(d_{j+i-i+1}-d_{j+i-i}) \\
&= \frac{1}{q^{j+i+1}}\left(v(a_1)+v(a_2)(-1+q-q^2+\ldots-q^{i-1})\right) \\
&\quad +\left(\frac{v(a_2)}{q^{j+i-i+1}}+d_{j+i-i-1}+1\right)-\left(\frac{v(a_1)}{q^{j+i-i}}+d_{j+i-i-1}+1+\delta_{j+i-i}\right) \\
&= \frac{1}{q^{j+i+1}}\left(v(a_1)(1-q^{i+1})+v(a_2)(-1+q-q^2+\ldots-q^{i-1}+q^i)\right)-\delta_{j+i-i} \\
&= \frac{1}{q^{j+i+1}}\left(v(a_1)(1-q^{i+1})+v(a_2)\frac{(-q)^{i+1}-1}{q+1}-q^{j+i+1}\delta_{j+i-i}\right) \\
&= \frac{q^{i+1}-1}{q^{j+i+1}}\left(\frac{v(a_2)}{q+1}-v(a_1)-\frac{q^{j+i+1}}{q^{i+1}-1}\delta_{j+i-i}\right),
\end{aligned}$$

hence, d_{j+i+1} has type I if and only if $\mathcal{D}_{j+i}<0$, which is if and only if $\frac{v(a_2)}{q+1}-v(a_1)<\frac{q^{j+i+1}\delta_{j+i-i}}{q^{i+1}-1}$, and we also see that the other cases follow as asserted. □

Corollary 2.5. *Let m be the smallest integer $m\geq 1$ such that $-v(j(\phi))\leq q^{m+1}$.*

(i) If d_{j+i} is of type III *and $-v(j(\phi))<q^{j+i+1}$ for some $j\geq m+1$ and $i\geq 1$, then d_{j+i+1} is of type* II.

For (ii) and (iii), suppose that $d_{j+1},d_{j+2},\ldots,d_{j+i}$ is a run of type II *for some $j\geq m$ and $i\geq 1$. Let $e=\frac{q^{i+j+1}}{q^{i+1}+1}$.*

(ii) If d_j is of type I*, then d_{j+i+1} is of type:*

- I *when i is even and $-v(j(\phi))>(q+1)e$, or when i is odd and $-v(j(\phi))<0$.*
- II *when i is even and $-v(j(\phi))<(q+1)e$, or when i is odd and $-v(j(\phi))>0$.*
- III *when i is even and $-v(j(\phi))=(q+1)e$, or when i is odd and $-v(j(\phi))=0$.*

(iii) If d_j is of type III*, then d_{j+i+1} is of type:*

- I *when i is odd and $v(j(\phi))>0$,*
- I *(if $\delta_j>0$) or* III *(if $\delta_j=0$) when i is odd and $v(j(\phi))=0$,*
- II *when i is even and $-v(j(\phi))<(q+1)e$*
- II *(if $\delta_j>0$) or* III *(if $\delta_j=0$) when i is even and $-v(j(\phi))=(q+1)e$*

(Note (i) and (iii) are only sufficient conditions.)

Proof. To show (i), since d_{j+i} is of type III, we have that

$$d_{j+i} \geq \frac{v(a_1)}{q^{j+i}} + 1 + d_{j+i-1}.$$

Thus,

$$\begin{aligned}\mathcal{D}_{j+i} &= \frac{v(a_1) - v(a_2)}{q^{j+i+1}} + d_{j+i} - d_{j+i-1} \\ &\geq \frac{v(a_1) - v(a_2)}{q^{j+i+1}} + \frac{v(a_1)}{q^{j+i}} + 1 \\ &= \frac{(q+1)v(a_1) - v(a_2) + q^{j+i+1}}{q^{j+i+1}},\end{aligned}$$

and $\mathcal{D}_{j+i} > 0$ implies that d_{j+i+1} is of type II. This happens if $-v(j(\phi)) < q^{j+i+1}$.

For (ii), we first note that $d_j = \frac{v(a_1)}{q^j} + d_{j-1} + 1$, and this is the case that $\delta_j = 0$ in (6). Therefore, (ii) follows from the proof of Lemma 2.4, where we note the index $j = m$ is now allowed.

Part (iii) follows from Lemma 2.4.

□

By Lemma 2.4 and Corollary 2.5 we can determine the following general behavior of the types of each d_j for $j \geq m+1$.

Proposition 2.6. *Let m be the smallest integer $m \geq 1$ such that $-v(j(\phi)) \leq q^{m+1}$.*

(i) If d_j is of type I *for $j \geq m+1$, then d_{j+1} is of type* II.

(ii) If d_{j-1} is of type I *and d_j is of type* II *for $j \geq m+1$, then d_{j+1} has type*

$$\begin{aligned} &\text{I} \quad \textit{if} \quad v(j(\phi)) > 0, \\ &\text{II} \quad \textit{if} \quad v(j(\phi)) < 0, \\ &\text{III} \quad \textit{if} \quad v(j(\phi)) = 0. \end{aligned}$$

(iii) Assume $v(j(\phi)) = 0$. If d_j, d_{j+1} has type III, II *respectively for any $j \geq m+1$, then d_{j+2} has type* I *or* III.

Proof. To prove (i), assume that d_j is of type I for $j \geq m+1$. Let

$$\mathcal{D}_j = \left(\frac{v(a_1)}{q^{j+1}} + d_j\right) - \left(\frac{v(a_2)}{q^{j+1}} + d_{j-1}\right),$$

then we have $\mathcal{D}_j = \frac{v(a_1)(q+1)-v(a_2)+q^{j+1}}{q^{j+1}}$, thus d_{j+1} has type II if and only if $\mathcal{D}_j > 0$ if and only if $-v(j(\phi)) < q^{j+1}$. Since $j > m$, we have $-v(j(\phi)) \leq q^{m+1} < q^{j+1}$, so d_{j+1} is of type II.

For (ii), assume that d_{j-1} is of type I and d_j is of type II for $j \geq m+1$. To determine the type of d_{j+1}, we apply (ii) of Corollary 2.5 (note the shift in indices $j = j-1$, $i = 1$) to deduce the type of d_{j+1} is I if $-v(j(\phi)) < 0$, and II if $-v(j(\phi)) > 0$, and III if $-v(j(\phi)) = 0$.

Part (iii) follows from (iii) of Corollary 2.5 (using the indices $j = j$ and $i = 1$).

□

In what follows, we show that the types of the sequence $d_1, d_2, \ldots, d_i, \ldots$ have four possible patterns.

Theorem 2.1. *Let $\phi_T = T + a_1\tau + a_2\tau^2$ be a Drinfeld A-module defined over C_∞ of rank 2. Let $e_\phi(z) = \sum_{i=0}^{\infty} c_i \tau^i(z)$ be its associated exponential function and let $d_i = v(c_i)/q^i$. We have the following cases for the types of the sequence $d_1, d_2, \ldots$*

Let m be the smallest integer $m \geq 1$ such that $-v(j(\phi)) \leq q^{m+1}$.

Case 1: $v(j(\phi)) > 0$.
Then the sequence $d_1, d_2, d_3, d_4, \ldots, d_j, \ldots$ has type I, II, I, II, …, *that is, d_j has type* I *if j is odd, and d_j has type* II *if j is even.*

Case 2: $v(j(\phi)) < 0$.
(i) If $0 < -v(j(\phi)) < q^{m+1}$, then $d_1, \ldots, d_m$ is a run of type I, *and d_{m+i} has type* II *for any $i \geq 1$.*

(ii) Assume $-v(j(\phi)) = q^{m+1}$. If there exists k such that k is the smallest integer ≥ 1 with $\delta_{m+(2k-1)} \neq \frac{q-1}{q^{2k}}$, then the sequence

$$d_1, \ldots, d_m, d_{m+1}, d_{m+2}, \ldots, d_{m+(2k-1)}, d_{m+2k}, d_{m+(2k+1)}, d_{m+(2k+2)}, d_{m+(2k+3)}, \ldots$$

has types I, …, I, III, II, …, III, II, I/II, II, II, …, *where d_1 through d_m have type* I *and d_{m+j} has the following types for $j \geq 1$:*

$$\begin{aligned} &\textit{type}\ \ \text{III} \quad \textit{if}\ \ j = 1, 3, 5, \ldots, (2k-1), \\ &\textit{type}\ \ \text{I or II} \quad \textit{if}\ \ j = 2k+1, \\ &\textit{type}\ \ \text{II} \quad \textit{otherwise}. \end{aligned}$$

The type of d_{m+2k+1} is I *if $\delta_{m+2k-1} > \frac{q-1}{q^{2k}}$ and* II *if $\delta_{m+2k-1} < \frac{q-1}{q^{2k}}$.*

If there is no such k, that is, $\delta_{m+(2k-1)} = \frac{q-1}{q^{2k}}$ for any $k \geq 1$, then the sequence

$$d_1, \ldots, d_m, d_{m+1}, d_{m+2}, \ldots, d_{m+(2k-1)}, d_{m+2k}, d_{m+(2k+1)}, d_{m+(2k+2)}, \ldots$$

has types $\mathrm{I}, \ldots, \mathrm{I}, \mathrm{III}, \mathrm{II}, \ldots, \mathrm{III}, \mathrm{II}, \mathrm{III}, \mathrm{II}, \ldots$, *where d_1 through d_m have type I and d_{m+j} has the following types for $j \geq 1$ with $k \geq 1$:*

$$\begin{aligned} &\textit{type III} \quad \textit{if} \quad j = 2k-1, \\ &\textit{type II} \quad \textit{if} \quad j = 2k. \end{aligned}$$

Case 3: $v(j(\phi)) = 0$.
The sequence $d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, \ldots$ has types:
$\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{II}$, I/III, II, I/III, II, ..., *where d_1 has type* I, *d_3 has type* III, *for $k \geq 1$, d_j has type* II *if $j = 2k$ and type* I *or* III *if $j = 2k+3$. Furthermore, if d_{2k+3} has type* I (*respectively,* III) *with $k \geq 1$, then d_{2k+5} has type* III (*respectively,* I *if $\delta_{2k+3} > 0$ or* III *if $\delta_{2k+3} = 0$*).

Proof. **Case 1:** $v(j(\phi)) > 0$. We know that d_1 has type I from its definition, and d_2 is of type II from Lemma 2.3 since $v(j(\phi)) > 0$. From $(i), (ii)$ of Proposition 2.6, the result follows.

(i) of Case 2: $0 < -v(j(\phi)) < q^{m+1}$. We know that d_{m+1} has type II from Lemma 2.3. Then d_{m+2} has type II by (ii) of Proposition 2.6. For the inductive hypothesis, we suppose that d_{m+j} has type II for $j \geq 1$. Now we determine the type of d_{m+j+1} by considering two cases: j is even or odd, by applying (ii) of Corollary 2.5 with the indices $j = m$ and $i = j$. If j is even, then d_{m+j+1} has type II if and only if $-v(j(\phi)) < (q+1)\frac{q^{m+j+1}}{q^{j+1}+1}$, which holds as $-v(j(\phi)) \leq q^{m+1} < (q+1)\frac{q^{m+j+1}}{q^{j+1}+1}$. If j is odd, then d_{m+j+1} is of type II as $-v(j(\phi)) > 0$.

(ii) of Case 2: $-v(j(\phi)) = q^{m+1}$. Then $d_1, \ldots, d_m$ is a run of type I and d_{m+1} is of type III. From (i) of Corollary 2.5 with indices $j = m$ and $i = 1$, since $-v(j(\phi)) = q^{m+1} < q^{m+2}$, d_{m+2} is of type II.

If $d_{m+(2k-1)}$ has type III and d_{m+2k} has type II for $k \geq 1$, then to determine the type of $d_{m+(2k+1)}$, we need to determine the sign of $\mathcal{D}_{m+2k}$. Since $d_{m+(2k-1)}$ has type III and d_{m+2k} has type II, we obtain that $\mathcal{D}_{m+2k} = \frac{q-1}{q^{2k}} - \delta_{m+(2k-1)}$. Thus, $d_{m+(2k+1)}$ has type I if $\delta_{m+(2k-1)} > \frac{q-1}{q^{2k}}$, type II if $\delta_{m+(2k-1)} < \frac{q-1}{q^{2k}}$ and type III if $\delta_{m+(2k-1)} = \frac{q-1}{q^{2k}}$.

First, assume that there exists the smallest integer $k \geq 1$ such that $\delta_{m+(2k-1)} \neq \frac{q-1}{q^{2k}}$. The assertion about the types of d_{m+j} for $1 \leq j \leq 2k$ then follows by choice of k. Now, $d_{m+(2k+1)}$ has types I or II and we show that d_{m+j} has type II for every $j > 2k+1$:

If $d_{m+(2k+1)}$ has type I, then by (i) of Proposition 2.6, $d_{m+(2k+2)}$ is of type II. Assuming that d_{m+j-1} has type II for $j \geq 2k+3$ with $d_{m+(2k+2)}, \ldots, d_{m+j-1}$ a run of type II, we claim that d_{m+j} has also type II. Using (ii) of Corollary 2.5 with indices $j = m+(2k+1)$ and $i = j-2k-2$, if j is odd (i.e. if i is odd), then d_{m+j} has type II since $-v(j(\phi)) > 0$, and if j is even (i.e. if i is even), then since $-v(j(\phi)) = q^{m+1} < (q+1)\frac{q^{m+j}}{q^{j-2k-1}+1}$, we have that d_{m+j} has type II.

Assume that $d_{m+(2k+1)}$ has type II, and assume $d_{m+2k}, d_{m+(2k+1)}, \ldots, d_{m+j-1}$ is a run of type II for $j \geq 2k+2$. If j is even, d_{m+j} has type II using (iii) of Corollary 2.5 with indices $j = m+(2k-1)$ and $i = j-2k$, since $-v(j(\phi)) < (q+1)\frac{q^{m+j}}{q^{j-2k+1}+1}$. If j is odd, by Lemma 2.4 with indices $j = m+(2k-1)$ and $i = j-2k$, then d_{m+j} has type II if and only if $\delta_{m+(2k-1)} < B := \frac{(q^{j-2k+1}-1)}{q^{j-1}(q+1)}$. Recall that $\delta_{m+(2k-1)} < \frac{q-1}{q^{2k}}$. In fact, $\frac{q-1}{q^{2k}} - B = \frac{1-q^{j-2k-1}}{q^{j-1}(q+1)} < 0$, so we have $\delta_{m+(2k-1)} < \frac{q-1}{q^{2k}} < B$. Thus d_{m+j} has type II.

Now, assume that there is no such k, that is, for any $k \geq 1$, $\delta_{m+(2k-1)} = \frac{q-1}{q^{2k}}$. Then $d_{m+(2k+1)}$ has a type III. Then $d_{m+(2k+2)}$ has type II by (i) of Corollary 2.5 using the indices $j = m+2k-1$ and $i = 2$, since $-v(j(\phi)) = q^{m+1} < q^{m+2k+2}$. The assertion thus follows as desired.

Case 3: $v(j(\phi)) = 0$. We then have $m = 1$, so d_1, d_2 have types I, II respectively. Using Corollary 2.5, d_3 has type III by (ii) with indices $j = 1$ and $i = 1$.

For any $j \geq 2$, if d_{2j-1} has type III, then by (i) of Corollary 2.5, d_{2j} has type II. Then by (iii) of Proposition 2.6, using indices $j = 2j-1$, d_{2j+1} has type I or III. Furthermore, d_{2j+1} has type I if $\delta_{2j-1} > 0$ and III if $\delta_{2j-1} = 0$ by using Lemma 2.4.

On the other hand, for any $j \geq 3$, if d_{2j-1} has type I, then d_{2j} has type II from (i) of Proposition 2.6. Thus d_{2j+1} has type III by (ii) of Proposition 2.6.

Combining both cases, we have that for $k \geq 1$, d_j has type II if $j = 2k$, type I or III if $j = 2k+3$. Furthermore, if d_{2k+3} has type I (respectively, III) with $k \geq 1$, then d_{2k+5} has type III (respectively, I if $\delta_{2k+3} > 0$, or III if $\delta_{2k+3} = 0$). □

The following corollary can be deduced from Theorem 2.1 by a somewhat lengthy calculation.

Corollary 2.7. *Under the notation and hypotheses of Theorem 2.1, we have the following.*

Case 1: $v(j(\phi)) > 0$. *Then*

$$v(c_i) = \begin{cases} v(a_2)\frac{q^i-1}{q^2-1} + \frac{i}{2}q^i & \text{if } 1 \le i \text{ is even,} \\ v(a_1) + v(a_2)\frac{q(q^{i-1}-1)}{q^2-1} + \frac{i+1}{2}q^i & \text{if } 1 \le i \text{ is odd.} \end{cases}$$

Note that $v(j(\phi)) = \infty$ *if and only if* $v(a_1) = \infty$. *Hence, if these conditions hold, then we have that* $v(c_i) = \infty$ *for odd* i, *that is* $c_i = 0$ *for odd* i. *Conversely, if* $\infty \ne v(j(\phi)) > 0$, *then all of the* $c_i \ne 0$.

Case 2: $v(j(\phi)) < 0$. *Let* m *be the smallest integer* $m \ge 1$ *such that* $v(j(\phi)) \ge -q^{m+1}$.

(i) Assume $v(j(\phi)) > -q^{m+1}$. *Then*

$$v(c_i) = \begin{cases} v(a_1)\frac{q^i-1}{q-1} + iq^i & \text{if } 1 \le i \le m \\ v(a_2)\frac{q^{j+1}-1}{q^2-1} + \frac{j+1}{2}q^{m+j} + \left(v(a_1)\frac{q^{m-1}-1}{q-1} + (m-1)q^{m-1}\right)q^{j+1} & \text{if } i = m+j, j \ge 1 \text{ is odd,} \\ v(a_2)\frac{q^j-1}{q^2-1} + \frac{j}{2}q^{m+j} + \left(v(a_1)\frac{q^m-1}{q-1} + mq^m\right)q^j & \text{if } i = m+j, j \ge 1 \text{ is even.} \end{cases}$$

In this case, note that $a_1 \ne 0$ *and* $c_i \ne 0$ *for all* i.

(ii) Assume $v(j(\phi)) = -q^{m+1}$. *Let* k *be the smallest integer* ≥ 1 *with* $\delta_{m+2k-1} \ne \frac{q-1}{q^{2k}}$ *(with the convention* $k = \infty$ *if such a* k *does not exist). Then*

$$v(c_i) = \begin{cases} v(a_1)\frac{q^i-1}{q-1} + iq^i & \text{if } 1 \le i \le m \\ v(a_2)\frac{q^{j+1}-1}{q^2-1} + \frac{j+1}{2}q^{m+j} & \\ \quad + \left(v(a_1)\frac{q^{m-1}-1}{q-1} + (m-1)q^{m-1}\right)q^{j+1} & \\ \quad + q^{m-1}(q-1)\frac{q^{j+1}-1}{q^2-1} & \text{if } i = m+j, 1 \le j < 2k-1 \text{ is odd,} \\ v(a_2)\frac{q^{j+1}-1}{q^2-1} + \frac{j+1}{2}q^{m+j} & \\ \quad + \left(v(a_1)\frac{q^{m-1}-1}{q-1} + (m-1)q^{m-1}\right)q^{j+1} & \\ \quad + q^{m+1}(q-1)\frac{q^{2k-2}-1}{q^2-1} + q^{m+2k-1}\delta_{m+2k-1} & \text{if } i = m+j, j = 2k-1, \\ v(a_1) + q^{m+2k+1} + qv(c_{m+2k}) & \text{if } i = m+2k+1, \delta_{m+2k-1} > \frac{q-1}{q^{2k}}, \\ v(a_2) + q^{m+2k+1} + q^2v(c_{m+2k-1}) & \text{if } i = m+2k+1, \delta_{m+2k-1} < \frac{q-1}{q^{2k}}, \\ v(a_2)\frac{q^j-1}{q^2-1} + \frac{j}{2}q^{m+2k+1+j} + v(c_{m+2k+1})q^j & \text{if } i = m+2k+1+j, 1 \le j \text{ is even }, \\ v(a_2)\frac{q^j-1}{q^2-1} + \frac{j}{2}q^{m+j} & \\ \quad + \left(v(a_1)\frac{q^m-1}{q-1} + mq^m\right)q^j & \text{if } i = m+j, 1 \le j \text{ is even.} \end{cases}$$

Case 3: $v(j(\phi)) = 0$. *Then*

$$v(c_i) = \begin{cases} v(a_1) + q & \text{if } i = 1, \\ v(a_2)\frac{q^i-1}{q^2-1} + \frac{i}{2}q^i & \text{if } 1 \leq i \text{ is even,} \\ v(a_1) + (1+\delta_i)\, q^i + qv(c_{i-1}) & \text{if } 1 < i \text{ is odd.} \end{cases}$$

3. Applications to zeroes of para-Eisenstein series

Let $\Omega = C_\infty - K_\infty$ be the Drinfeld upper half plane. By uniformization, the set $\mathrm{GL}_2(A)\backslash\Omega$ is in bijection with the C_∞-isomorphism classes of Drinfeld A-modules. By uniformization, we may regard the coefficients $c_i = c_i(z)$ as functions on the upper-half plane Ω (i.e. by taking the exponential function of the Drinfeld A-module corresponding to the lattice $A + Az$), dubbed *the para-Eisenstein series* by Gekeler [3], and studied in [2, 5].

Let

$$\mathcal{F} = \{z \in C_\infty : |z| = |z|_i \geq 1\},$$
$$\mathcal{F}_k = \left\{z \in C_\infty : |z| = |z|_i = q^k\right\},$$

where $k \geq 0$ and $|z|_i = \inf_{x \in K_\infty} |z - x|$.

The subset $\mathcal{F}$ of Ω is a kind of fundamental domain for Ω under the action of $\Gamma = \mathrm{GL}_2(A)$. For instance, it has the property that each element $z \in \Omega$ is Γ-equivalent to an element of $\mathcal{F}$. Further properties can be found in [2, Corollary 6.7], in particular an explicit description of the isotropy subgroups.

The following corollary shows that we can derive $v(c_i(z))$ explicitly in terms of q, m for $z \in \mathcal{F}_m$ with $m \geq 1$ and $i = m + j$ with j even ≥ 2. This case is not covered in [2].

Corollary 3.1. *Let $z \in \mathcal{F}_m$ with $m \geq 1$ and $i = m + j$ with j even ≥ 2. Then the formula for $v(c_i(z))$ is given as follows:*

$$v(c_i(z)) = \left(-q^2 - q + q^{1+m}\right)\frac{q^j - 1}{q^2 - 1} + \frac{j}{2}q^{m+j} + \left(-q\frac{q^m - 1}{q - 1} + mq^m\right)q^j.$$

Proof. Substituting the values of $v(a_1(z))$ and $v(a_2(z))$ given by [4, Corollary 2.18, Theorem 2.13] into the formula for $v(c_i(z))$ given by Case 2 (ii) of Corollary 2.7 yields the desired result. □

For $z \in \mathcal{F}_0$, let $j(z)$ be the j-invariant of the Drinfeld A-module corresponding to the lattice $A + Az$. The following theorem is case $k = 0$ of [4, Theorem 2.13] (for a different proof, see [1, §3]).

Theorem 3.1. *Let $z \in \mathcal{F}_0$. Then*

$$v(a_1(z)) = \frac{v(j(z)) - q^2}{q+1}$$
$$v(a_2(z)) = -q^2.$$

Thus, the valuations $v(c_i(z))$ for $z \in \mathcal{F}_0$ are determined entirely and explicitly in terms of $v(j(z)) \geq -q$ by Corollary 2.7, except in the situation when $v(j(z)) = 0$ and $i > 1$ is odd.

The quantity $j(z)$ with z being a zero of c_i is called a *j-zero* of c_i.

Theorem 3.2. *For $z \in \mathcal{F}$ we have the following*

- $v(j(z)) \geq -q$ *if and only if* $z \in \mathcal{F}_0$
- $v(j(z)) = -q^{k+1}$ $(k \geq 1)$ *if and only if* $z \in \mathcal{F}_k$.

Proof. [4, Theorem 2.17] [5, Corollary 3.11]. □

Let $\delta(i) = 1$ if i is odd and $\delta(i) = 0$ if i is even.

Theorem 3.3. *For $0 \leq k < \frac{i-\delta(i)}{2}$, there are precisely q^{2k} j-zeroes x of c_i that satisfy $v(x) = -q^{i-2k}$. For i odd, there are $\lambda(i) = \frac{q^{i-1}-1}{q+1}$ further j-zeroes x with $v(x) = 0$ and the j-zero $x = 0$. These are all the j-zeroes of c_i.*

Proof. [2, Theorem 8.11]. □

Theorem 3.4. *All the zeroes of c_i in $\mathcal{F}$ lie in some $\mathcal{F}_k$. If i is even (odd), c_i has precisely $(q-1)q^i$ zeroes in $\mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5, \ldots, \mathcal{F}_{i-1}$ (in $\mathcal{F}_0, \mathcal{F}_2, \mathcal{F}_4, \ldots, \mathcal{F}_{i-1}$) each, and no further zeroes in $\mathcal{F}$.*

Proof. [2, Corollary 8.12]. □

Lemma 3.2. *Let $z_0 \in \Omega$ and $j(z_0) \neq 0$. If $c_i(z_0) = 0$, then $d_i = d_i(z_0)$ is of type III.*

Proof. The hypothesis that $j(z_0) \neq 0$ implies that $d_1 = \frac{v(a_1)}{q} + 1 + d_0 \neq \infty$.

If d_i is of type I, then we know that

$$d_i = \frac{v(a_1)}{q^i} + d_{i-1} + 1 < \frac{v(a_2)}{q^i} + d_{i-2} + 1. \tag{7}$$

If $c_{i-1}(z_0) \neq 0$, then we know that $d_{i-1} \neq \infty$, and hence $d_i \neq \infty$ so $c_i(z_0) \neq 0$. On the other hand, if $c_{i-1}(z_0) = 0$, then $d_{i-1} = \infty$, contradicting the inequality (7) that shows that d_i is of type I.

If we assume that d_i is of type II, then we still get a contradiction by a similar argument as the type I case above. The result thus follows. $\square$

Theorem 3.5. *If $z_0 \in \mathcal{F}_m$, $m \geq 1$, and $c_i(z_0) = 0$, then $c_j(z_0) \neq 0$ for $i \neq j$. That is, any two $c_i(z)$ and $c_j(z)$, $i \neq j$, do not have any common zeroes on $\cup_{k\geq 1}\mathcal{F}_k$.*

Proof. The condition on z_0 shows that $-v(j(\phi)) = \log_q |j(z_0)| = q^{m+1}$ for some $m \geq 1$ so we are in Case 2 (ii) of Theorem 2.1. Thus, $j(z_0) \neq 0$. By Lemma 3.2, the type of d_i is III. Moreover, by Corollary 2.7, we see the only coefficients with $c_i(z_0) = 0$ are when $i = m + 2k - 1$, $i = m + 2k + 1$, or $i = m + 2k + 1 + j$, where $j \geq 2$ is even, as these are the only type III coefficients with undetermined valuation. From Case 2 (ii) of Corollary 2.7, $c_{m+2k+1} = 0$ only happens for the following two possible cases:

(i) $c_{m+2k} = 0$ and $\delta_{m+2k-1} > \frac{q-1}{q^{2k}}$
(ii) $c_{m+2k-1} = 0$ and $\delta_{m+2k-1} < \frac{q-1}{q^{2k}}$

But, in the first case, $v(c_{m+2k}) = \infty$, and it only happens if $d_{m+2k} = \infty$, which is false from the formula for $v(c_{m+n})$ with n a positive even in Corollary 2.7. In the second case, $v(c_{m+2k-1}) = \infty$, and it only happens if $d_{m+2k-1} = \infty$, hence $\delta_{m+2k-1} = \infty$; this is a contradiction to $\delta_{m+2k-1} < \frac{q-1}{q^{2k}}$. Finally, $c_{m+2k+1+j} = 0$ for $j \geq 2$ even only if $c_{m+2k+1} = 0$, which we already ruled out. Thus, if $c_i(z_0) = 0$, then it must be the case that $i = m + 2k - 1$. $\square$

If $z \in \mathcal{F}_0$, then $v(j(z)) \geq -q$. From Corollary 2.7 (Case 1, Case 2(i), Case 3), it follows that if $v(j(z)) > -q, v(j(z)) \neq 0$, then $c_i(z) \neq 0$ for all $i \geq 1$, unless $j(z) = 0$, in which case $c_i(z) = 0$ for all odd $i \geq 1$ and $c_i(z) \neq 0$ for all even $i \geq 2$. If $v(j(z)) = 0$, then none of the $c_i(z) = 0$ for $i \geq 2$ even, and if $c_i(z) = c_j(z) = 0$, then i and j are odd. It appears that the question of common zeroes in this final case requires further techniques beyond the scope of this paper.

4. Some concluding remarks

In [2], one uses the recursion formulae for $c_i(z)$ to determine the number of j-zeroes of $c_i(z)$ and their valuations. This information is vertical in the sense that one studies $c_i(z)$ for fixed i. In comparison, our methods are horizontal in the sense that z is fixed and we consider the relationship between the $c_i(z)$'s.

The valuations $v(c_i(z))$ for even i are also determined for $z \in \mathcal{F}_0$ in [2]. On the other hand, for odd i and $z \in \mathcal{F}_0$, the methods in [2] determine only the spectral norm of $c_i(z)$. In comparison, our methods can determine the exact valuation $v(c_i(z))$ for odd i in some cases.

For $z \in \mathcal{F}$ but $z \notin \mathcal{F}_m$ for any $m \geq 0$, the interpolation property can be exploited to determine the valuations of $c_i(z)$ for both odd and even i as described in [2], though not all of the possible results are tabulated in loc. cit.

In fact, from Corollary 2.7 and Theorem 3.5, we can recover some of results of Gekeler [2] on the distribution of zeroes of $c_i(z)$. From Case 1 of Corollary 2.7, we get that 0 is a j-zero of $c_i(z)$ when i is odd; compare with [2, Theorem 8.11]. From the proof of Theorem 3.5, we get that if $z_0 \in \mathcal{F}_m$ is a zero of c_i with $m \geq 1$, then m is odd with $m \leq i-1$ (resp. even $m \leq i-1$) if i is even (resp. odd); compare with [2, Corollary 8.12].

As shown in [2, Corollary 8.13], the norm $|c_i(z)|$ is constant on $\cup_{m \geq i} \mathcal{F}_m$ and $v(c_i(z)) = iq^i - (\frac{q^i-1}{q-1})q$. This case is also covered by Corollary 2.7, Case 2(ii) when $1 \leq i \leq m$, and corresponds to the behavior on Ω at infinity.

We note that it is necessary to calibrate the valuations obtained in [2] in relation to the defining coefficients a_i of the Drinfeld A-module ϕ, as the assumption in loc. cit. is that the lattice associated to ϕ is of the form $\Lambda = A + Az$ (so the valuations obtained there are only valid after taking an appropriate twist of the originally given ϕ). This requires further arguments (see [1, Corollary 3.1]).

Another interesting property of our method is that the case $z \in \mathcal{F}_0$ if and only if $-v(j(\phi)) \leq q$ separates into three cases, $-v(j(\phi)) < 0, -v(j(\phi)) = 0, 0 < -v(j(\phi)) \leq q$, whereas the methods in [2] do not seem to immediately lend themselves to a more detailed analysis of the case $-v(j(\phi)) \leq q$. It would be interesting to investigate this further. Some additional elaboration of subsequent recent work by Gekeler [5] on explicit formulae for the j-function may provide a conceptual elucidation of our findings.

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